Expected Utility

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Cornell University · Decision Theory · Spring 2017

motivating examples

powerball


POWERBALL
TEXAS LOTTERY
MEGA MILLIONS
$ 700 MILLION
$ 165 MILLION
LAMAR
REGULAR
DIESEL
1.89

powerball

POWERBALL EXPECTED PAYOUT

NUMBERS MATCHED	PRIZE	PRIZE - COST	LIKELIHOOD	PROBABILITY	(PRIZE - COST) X PROBABILITY
5 white and red	$450,000,000	$449,999,998	1 in 292,201,338	0.0000000034	$1.54
5 white	$1,000,000	$999,998	1 in 11,688,053.52	0.0000000856	$0.09
4 white and red	$50,000	$49,998	1 in 913,129.18	0.0000010951	$0.05
4 white	$100	$98	1 in 36,525.17	0.0000273784	$0.00
3 white and red	$100	$98	1 in 14,494.11	0.0000689935	$0.01
3 white	$7	$5	1 in 579.76	0.0017248517	$0.01
2 white and red	$7	$5	1 in 701.33	0.0014258623	$0.01
1 white and red	$4	$2	1 in 91.98	0.0108719287	$0.02
Red	$4	$2	1 in 38.32	0.0260960334	$0.05
Nothing	$0	-$2	1 in 1.04	0.9597837679	-$1.92

EXPECTED VALUE: -$0.14

SOURCE: Business Insider calculations with odds from powerball.com

BUSINESS INSIDER

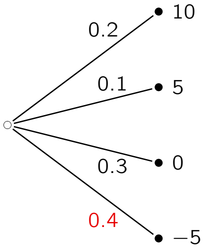
- Flip a fair coin until it lands tails
- If we flipped the coin n times, you get \$$2^n$
- How much would you be willing to pay to participate?

$$\mathbb{E}[\,2^{\mathrm{n}}\,] = \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 4 + \frac{1}{8}\cdot 8 + \ldots = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot 2^n = \infty$$

$$\mathbb{E}[\,\log(2^{\mathrm{n}})\,] = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\log\left(2^n\right) = \log(2)\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot n = 2\log(2) \approx 0.60$$

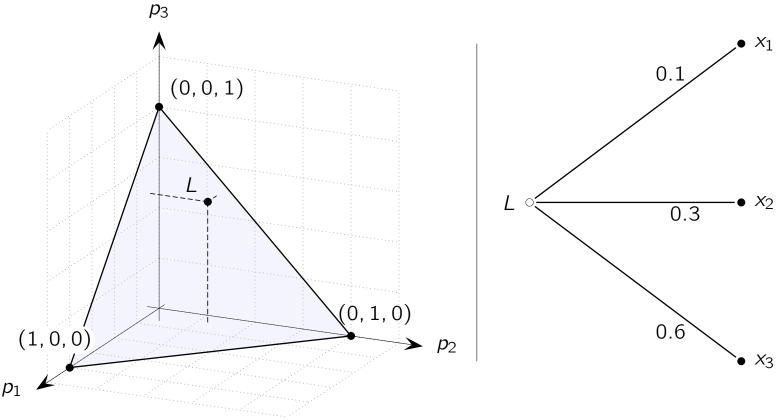
von Neumann and Morgestern

- A simple lottery is a tuple $L = (p_1, x_1;\ p_2, x_2;\ \ldots p_n, x_n)$
  - Monetary prizes $x_1, \ldots, x_n \in X \subseteq \mathbb{R}$
  - Probability distribution $(p_1, \ldots, p_n)$, p_i is the probability of x_i
- Let $\mathcal{L}$ denote the set of simple lotteries
- **Example:** $L = (10, 0.2;\ 5, 0.1;\ 0, 0.3;\ -5, 0.4)$



- Simple lotteries given a fixed set of prizes $x_1, \ldots, x_n$ correspond to points in the n-dimensional simplex

$$\Delta^n = \left\{ (p_1, \ldots p_n) \in \mathbb{R}^n \;\middle|\; 0 \leq p_i \leq 1 \quad \& \quad p_1 + \ldots + p_n = 1 \right\}$$



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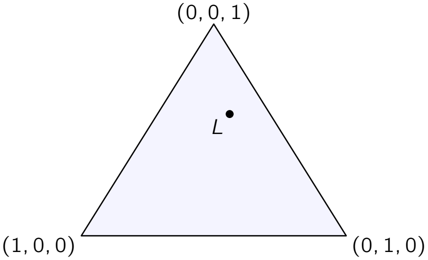
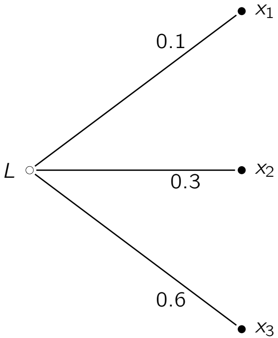




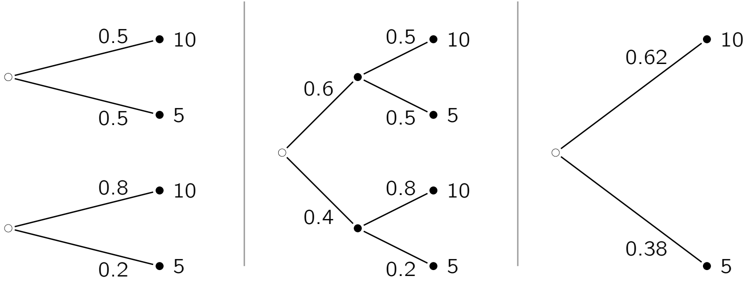
- For $0 \le \alpha \le 1$ and lotteries $L = (p_1, x_1;\ p_2, x_2;\ \dots p_n, x_n)$ and $M = (q_1, x_1;\ q_2, x_2;\ \dots q_n, x_n)$ with the same set of prizes, define

$$\alpha L \oplus (1-\alpha)M = \big(r_1, x_1;\ r_2, x_2;\ \dots r_n, x_n\big)$$

where

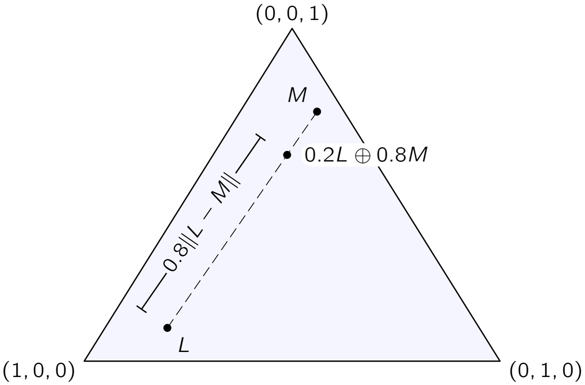
$$r_k = \alpha p_k + (1-\alpha) q_k$$

- **Example:** $L = (0.5, 10;\ 0.5, 5)$, $M = (0.8, 10;\ 0.2, 5)$, $\alpha = 0.6$

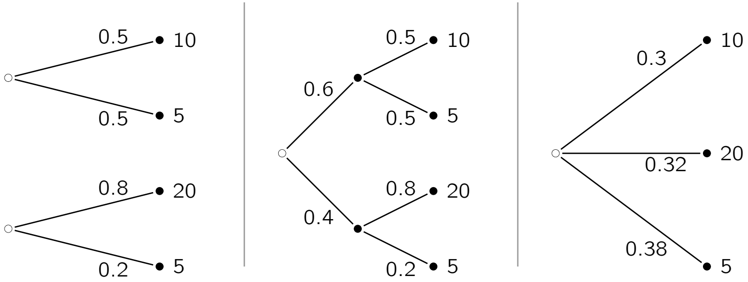


geometry of mixtures

- With a fixed set of prizes $x_1, \ldots, x_n$, mixtures between lotteries correspond to points in the line segment between them
- The mixture weights determine the location within the segment



- Also possible to mix lotteries with different prizes
- **Example:** $L = (0.5, 10;\ 0.5, 5)$, $M = (0.8, 20;\ 0.2, 5)$, $\alpha = 0.6$



$$\alpha L \oplus (1-\alpha)M = (0.3, 10;\ 0.32, 20;\ 0.38, 5)$$

- Reported preferences $\succ$ on $\mathcal{L}$
- A utility function $U : \mathcal{L} \to \mathbb{R}$ for $\succ$ is an expected utility function if it can be written as

$$U(L) = \sum_{k=1}^{n} p_i u(x_i)$$

for some function $u : \mathbb{R} \to \mathbb{R}$

- If you think of the prizes as a random variable $\mathbf{x}$, then

$$U(L) = \mathbb{E}_L \left[u(\mathbf{x}) \right]$$

- The function u is called a Bernoulli utility function

expected utility axioms

- **Axiom 1**: (Preference order) $\succ$ is a asymmetric and negatively transitive
- **Axiom 2**: (Continuity) For all simple lotteries $L, M, N \in \mathcal{L}$, if $L \succ M \succ N$ then there exist $\alpha, \beta \in (0, 1)$ such that

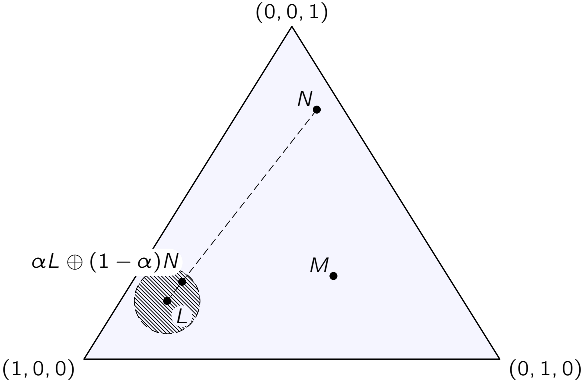
$$\alpha L \oplus (1-\alpha)N \succ M \succ \beta N \oplus (1-\beta)N$$

- **Axiom 3**: (Independence) For all lotteries $L, M, N \in \mathcal{L}$ and $\alpha \in (0, 1]$, if $L \succ M$, then

$$\alpha L \oplus (1-\alpha)N \succ \alpha M \oplus (1-\alpha)N$$

continuity

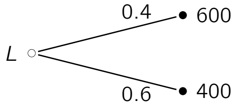
- The continuity axiom can be thought of as requiring that strict preference is preserved by sufficiently small perturbations in the probabilities
  - If $L \succ M$, then so are lotteries which are close enough to L (hatched area)
  - This includes $\alpha L \oplus (1-\alpha)N$ with α close enough to 1

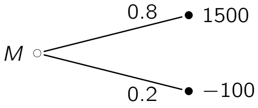


- If L is preferred to M, then a mixture of L with N is also preferred to a mixture of M with N using the same mixing weights
- Independence gives the expected-utility structure
- Similar to the independent-factors requirement from previous notes (expected utility is a form of additive separability)

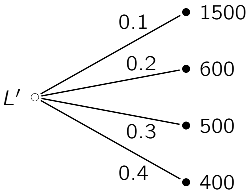
example

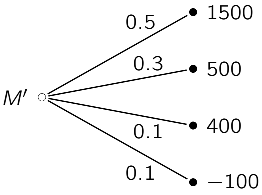
- How do you rank the following lotteries?





- How do you rank the following lotteries?

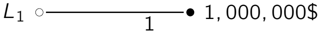


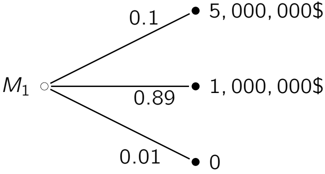


- Independence says that if you prefer L to M, then you also prefer L' to M'
- Note that $L' = 0.5L \oplus 0.5N$ and $M' = 0.5M \oplus 0.5N$, for some lottery N (which lottery?)

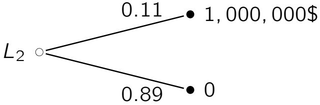
allais' paradox

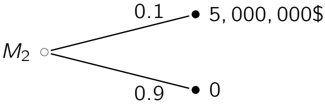
- How do you rank the following lotteries?





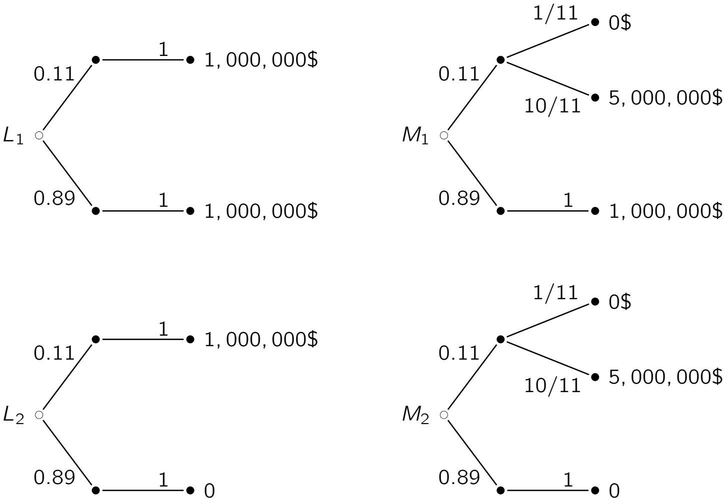
- How do you rank the following lotteries?





- Many people report $L_1 \succ M_1$ and $M_2 \succ L_2$

- Note that we can write



- Independence would imply that $L_1 \succ M_1$ if and only if $L_2 \succ M_2$ (why?)

Theorem:

(a) A binary relation $\succ$ over $\mathcal{L}$ has an expected utility representation if and only if it satisfies axioms 1–3

(b) If U and V are expected utility representations of $\succ$, then there exist constants $a, b \in \mathbb{R}$, $a > 0$, such that $U(\cdot) = a \cdot V(\cdot) + b$

- Suppose U is an expected utility representation of $\succ$
- Axiom 1 follows from the same arguments as before
- For $0 \leq \alpha \leq 1$ and lotteries $L = (p_1, x_1;\ p_2, x_2;\ \dots p_n, x_n)$ and $M = (q_1, x_1;\ q_2, x_2;\ \dots q_n, x_n)$ note that

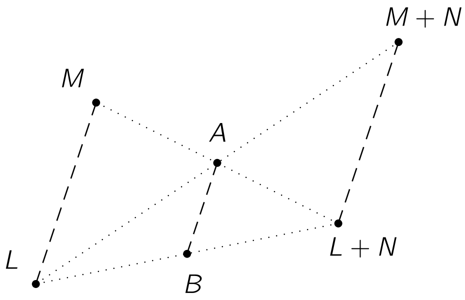
$$
\begin{aligned}
V\big(\alpha L \oplus (1-\alpha)M\big) &= \sum_{i=1}^{n} \big[\alpha p_i + (1-\alpha)q_i\big] \cdot u(x_i) \\
&= \sum_{i=1}^{n} \big[\alpha p_i u(x_i) + (1-\alpha) q_i u(x_i)\big] \\
&= \alpha \sum_{i=1}^{n} p_i u(x_i) + (1-\alpha) \sum_{i=1}^{n} (q_i u(x_i) \\
&= \alpha V(L) + (1-\alpha) V(M)
\end{aligned}
$$

- From here, it is straightforward to show that $\succ$ satisfies axioms 2 & 3

- Fix the set of prizes so that lotteries can be though of as vectors in Δ^n
- The following proposition that, under axioms 1–3, preferences are preserved under translations
- This means that the indifference curves are parallel lines

Proposition: Given lotteries $L, M \in \Delta^n$, and a vector $N \in \mathbb{R}^n$, if $L + N$ and $M + N$ are also lotteries and $\succ$ satisfies axioms 1–3, then

$$L \succ M \qquad \Leftrightarrow \qquad (L + N) \succ (M + N)$$



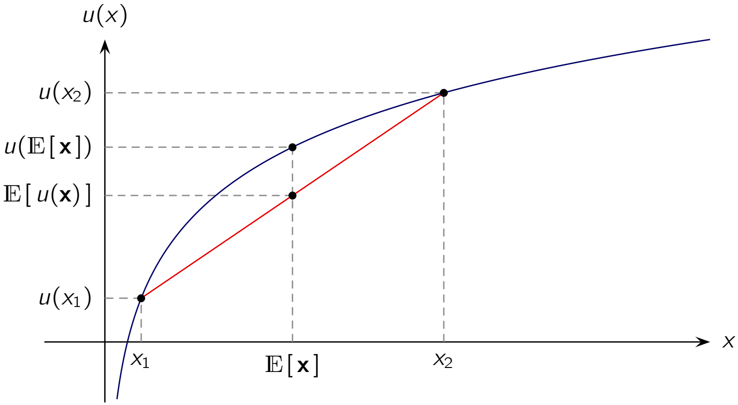
Proof sketch:

- If $(L + N)$ and $(M + N)$ are lotteries, then so are A and B
- $A = 0.5M \oplus 0.5(L + N)$ and $A = 0.5L \oplus 0.5(M + N)$
- Since $A = 0.5M \oplus 0.5(L + N)$, independence says that if $L \succ M$ then $B \succ A$
- Since $A = 0.5L \oplus 0.5(M + N)$, independence says that if $B \succ A$ then $(L + N) \succ (M + N)$

risk aversion

- For the rest of these slides, suppose u is strictly increasing (our decision maker always prefers more money) and twice continuously differentiable
- Risk-neutral decision maker – $\mathbb{E}[\, u(\mathbf{x})\,] = u(\mathbb{E}[\,\mathbf{x}\,])$ for every random variable $\mathbf{x}$
- Risk-averse decision maker – $\mathbb{E}[\, u(\mathbf{x})\,] \leq u(\mathbb{E}[\,\mathbf{x}\,])$ for every r.v. $\mathbf{x}$
- Risk-loving decision maker – $\mathbb{E}[\, u(\mathbf{x})\,] \geq u(\mathbb{E}[\,\mathbf{x}\,])$ for every r.v. $\mathbf{x}$

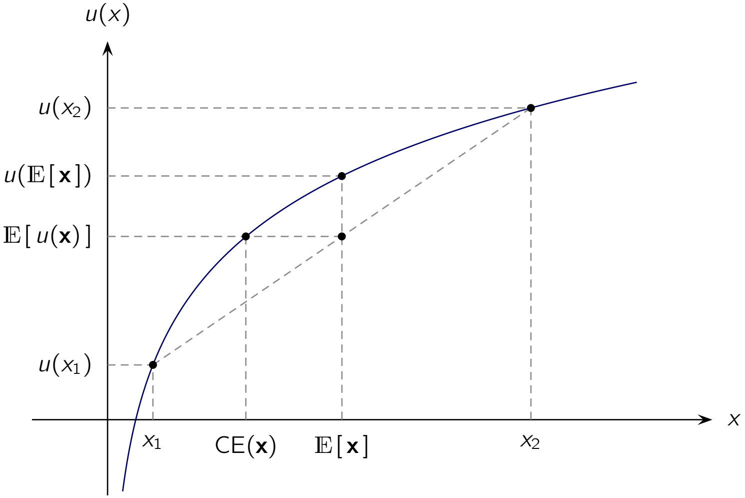
- A set is convex if it contains all the line-segments between its points
- A function is concave if its hypograph is a convex set
- Risk aversion is equivalent to u being concave



Definition: Given u, he certainty equivalent of a lottery $\mathbf{x}$ is the is the guaranteed amount of money that an individual with Bernoulli utility function u would view as equally desirable as $\mathbf{x}$, i.e.,

$$CE_u(\mathbf{x}) = u^{-1}\left(\mathbb{E}[\,u(\mathbf{x})\,]\right)$$

- Risk-neutral decision maker – $CE(L) = \mathbb{E}[\,\mathbf{x}\,]$ for every r.v. $\mathbf{x}$
- Risk-averse decision maker – $CE(L) \leq \mathbb{E}[\,\mathbf{x}\,]$ for every r.v. $\mathbf{x}$
- Risk-loving decision maker – $CE(L) \geq \mathbb{E}[\,\mathbf{x}\,]$ for every r.v. $\mathbf{x}$


$u(x)$
$u(x_2)$
$u(\mathbb{E}[\mathbf{x}])$
$\mathbb{E}[u(\mathbf{x})]$
$u(x_1)$
x
x_1
$\mathrm{CE}(\mathbf{x})$
$\mathbb{E}[\mathbf{x}]$
x_2

Definition: The arrow-prat coefficient of absolute risk aversion of u at x is

$$A_u(x) = -\frac{u''(x)}{u'(x)}$$

- Constant absolute risk aversion (CARA)

$$u(x) = -\exp(-\alpha x)$$

- Indeed $u'(x) = \alpha u(x)$ and $u''(x) = \alpha^2 u(x) \Rightarrow A_u(x) = \alpha$

more risk averse than

Theorem: Given any two strictly increasing Bernoulli utility functions u and v, the following are equivalent

(a) $A_u(x) \geq A_v(x)$ for all x

(b) $CE_u(\mathbf{x}) \leq CE_v(\mathbf{x})$ for all $\mathbf{x}$

(c) There exists a strictly increasing concave function g such that $u = g \circ v$

- In that case, we say that v is (weakly) more risk averse than u

- There always exist a strictly increasing and twice continuously differentiable function g such that $v = g \circ u$ (why?)
- By the chain-rule of differential calculus

$$v'(x) = g'(u(x))u'(x)$$

$$v''(x) = g'(u(x))u''(x) + g''(u(x))(u'(x))^2$$

- If g is concave, then $g'' < 0$ and thus

$$A_v(x) = -\frac{v''(x)}{v'(x)} = -\frac{g'(u(x))u''(x) + g''(u(x))(u'(x))^2}{g'(u(x))u'(x)}$$

$$= A_u(x) - \frac{g''(u(x))u'(x)}{g'(u(x))} \geq A_u(x)$$

proof sketch

- If g is concave, then Jensen's inequality implies that

$$\begin{aligned} v(CE_v(\mathbf{x})) &= \mathbb{E}[\, v(\mathbf{x})\,] = \mathbb{E}[\, g(u(\mathbf{x}))\,] \\ &\leq g\left(\mathbb{E}[\, u(\mathbf{x})\,]\right) = g\left(u(CE_u(\mathbf{x}))\right) = v(CE_u(\mathbf{x})) \end{aligned}$$

- Since v is strictly increasing, this implies that

$$CE_v(\mathbf{x}) \leq CE_u(\mathbf{x})$$

optimal portfolios

- An expected utility maximizer with initial wealth ω must decide a quantity α to invest on a risky asset

- The asset has a random gross return of $\mathbf{z}$ per dollar invested
- The final wealth of the investor will be $w - \alpha + \alpha\mathbf{z}$

- The optimal investment is the solution to the program

$$\begin{aligned} \max_{\alpha} \quad & \mathbb{E}\left[\, u\big(w + \alpha(\mathbf{z} - 1)\big) \,\right] \\ \text{s.t.} \quad & 0 \leq \alpha \leq w \end{aligned}$$

- Let α^* denote this solution

Proposition: A risk averse agent will always invest a positive amount on assets with positive expected return, i.e., if $\mathbb{E}[\,\mathbf{z}\,] > 1$ then $\alpha^* > 0$

Proof:

- Let $U(\alpha)$ denote the agent's expected utility

$$U'(\alpha) = \mathbb{E}\left[\,(\mathbf{z}-1)u'\big(w + \alpha(\mathbf{z}-1)\big)\,\right]$$

- If $\mathbb{E}[\,\mathbf{z}\,] > 1$, then U is strictly increasing at 0 because

$$U'(0) = \mathbb{E}[\,(\mathbf{z}-1)u'(w)\,] = u'(w)\big(\mathbb{E}[\,\mathbf{z}\,] - 1\big) > 0$$

- Suppose there are two assets with i.i.d. returns $\mathbf{z}_1$ and $\mathbf{z}_2$
- The investor chooses investments $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 \leq q$
- Let $U(\alpha_1, \alpha_2)$ denote the investor's expected utility

$$U(\alpha_1, \alpha_2) = \mathbb{E}\left[\, u\big(w + \alpha_1(\mathbf{z}_1 - 1) + \alpha_2(\mathbf{z}_2 - 1)\big) \,\right]$$

Proposition: A risk averse agent will always diversify among risky i.i.d. assets with positive returns, i.e., if $\mathbb{E}[\,\mathbf{z}_i\,] > 1$ and $\mathbb{V}[\,\mathbf{z}_i\,] > 0$, then $\alpha_1^* > 0$ and $\alpha_2^* > 0$.

- We already know that the optimal portfolio cannot be $(0, 0)$ (why?)
- For any portfolio without diversification $(\alpha^0, 0)$ we have that

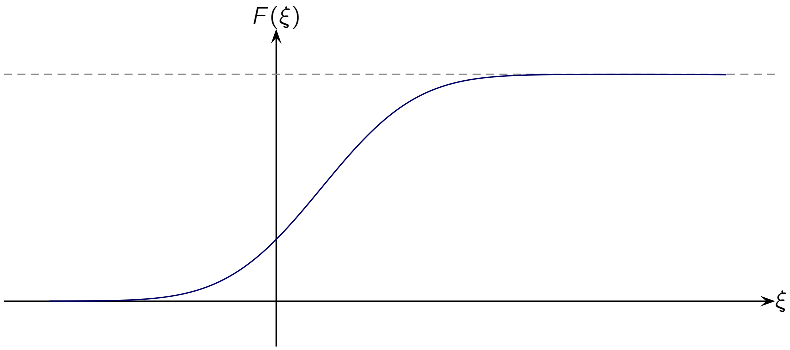
$$\begin{aligned}
U(\alpha^0, 0) &= \frac{1}{2}\mathbb{E}\left[\, u\big(w + \alpha^0(\mathbf{z}_1 - 1)\big) \,\right] + \frac{1}{2}\mathbb{E}\left[\, u\big(w + \alpha^0(\mathbf{z}_2 - 1)\big) \,\right] \\
&= \mathbb{E}\left[\, \frac{1}{2}u\big(w + \alpha^0(\mathbf{z}_1 - 1)\big) + \frac{1}{2}u\big(w + \alpha^0(\mathbf{z}_2 - 1)\big) \right] \\
&< \mathbb{E}\left[\, u\left(\frac{1}{2}\big(w + \alpha^0(\mathbf{z}_1 - 1)\big) + \frac{1}{2}\big(w + \alpha^0(\mathbf{z}_2 - 1)\big)\right) \right] \\
&= \mathbb{E}\left[\, u\left(w + \frac{1}{2}\alpha^0(\mathbf{z}_1 - 1) + \frac{1}{2}\alpha^0(\mathbf{z}_2 - 1)\right) \right] \\
&= U\left(\frac{1}{2}\alpha^0,\ \frac{1}{2}\alpha^0\right)
\end{aligned}$$

comparing distributions

- The cumulative distribution functions (c.d.f.) of a random variable $\mathbf{x}$ is the function $F : \mathbb{R} \to [0,1]$ given by

$$F(\xi) = \Pr(\mathbf{x} \leq \xi)$$

- C.d.f.s are non-decreasing, left-continuous, satisfy $\lim_{\xi \to -\infty} F(\xi) = 0$ and $\lim_{\xi \to \infty} F(\xi) = 1$



- Consider random variables **x** and **y** with c.d.f.s F and G
- That is $F(\xi) = \Pr(\mathbf{x} \leq \xi)$ and $G(\xi) = \Pr(\mathbf{y} \leq \xi)$

- When can we say that **x** is “greater” than **y**?
  - $\mathbb{E}[\mathbf{x}] > \mathbb{E}[\mathbf{y}]$ is probably not enough
  - $\min\{\text{support}(\mathbf{x})\} > \max\{\text{support}(\mathbf{y})\}$ is probably too much

- When can we say that **x** is “riskier” than **y**?
  - $\mathbb{V}[\mathbf{x}] > \mathbb{V}[\mathbf{y}]$ is probably not enough

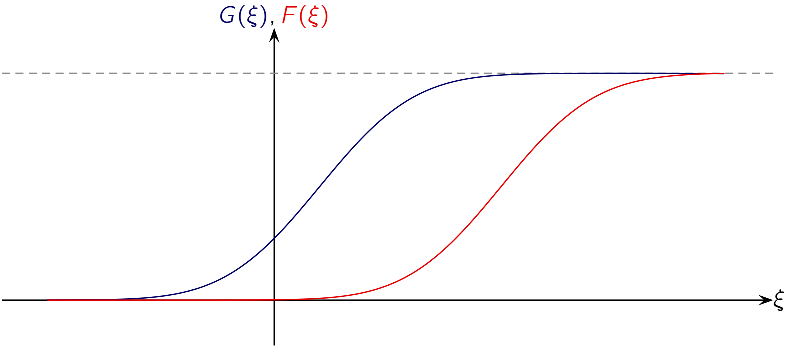
- Say that F first-order stochastically dominates G if every expected utility maximizer with monotone preferences would choose **x** over **y**

Definition: Say that $F \succ_{\text{FOSD}} G$ if for every non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ we have that $\mathbb{E}[\, u(\mathbf{x}) \,] \geq \mathbb{E}[\, u(\mathbf{y}) \,]$

- First-order stochastic dominance can be characterized in terms of distribution functions
- The following proposition asserts that $\mathbf{x} \succ_{\text{FOSD}} \mathbf{y}$ if for every number ξ, **y** taking a value smaller than ξ is more likely than **x** taking a value smaller than ξ

Proposition: $\mathbf{x} \succ_{\text{FOSD}} \mathbf{y}$ if and only if $F(\xi) \leq G(\xi)$

first order stochastic dominance



$$F \succ_{\text{FOSD}} G$$

- Suppose $F(\xi) > G(\xi)$ for some ξ
  - Let $u : \mathbb{R} \to \mathbb{R}$ be the Bernoulli utility function

$$u(x) = \begin{cases} 1 & \text{if } x > \xi \\ 0 & \text{otherwise} \end{cases}$$

  - Then $\mathbb{E}[\, u(\mathbf{x})\,] = 1 - F(\xi) < 1 - G(\xi) = \mathbb{E}[\, u(\mathbf{y})\,]$

- Suppose $F(\xi) \leq G(\xi)$ for all ξ and u, F and G are differentiable
  - Integrating by parts:

$$\mathbb{E}[\, u(\mathbf{x})\,] = -\int_{-\infty}^{\infty} u'(\xi) F(\xi)\, d\xi$$

  - Therefore

$$\mathbb{E}[\, u(\mathbf{x})\,] - \mathbb{E}[\, u(\mathbf{y})\,] = -\int_{-\infty}^{\infty} u'(\xi)\big(F(\xi) - G(\xi)\big)\, d\xi \geq 0$$

second order stochastic dominance

- First-order stochastic dominance is a very incomplete ranking
- More comparisons if we further restrict the set of utility functions
- Say that F second-order stochastically dominates G if every expected utility maximizer with monotone and concave preferences would choose $\mathbf{x}$ over $\mathbf{y}$

> **Definition:** Say that $F \succ_{\mathrm{SOSD}} G$ if for every non-decreasing and concave function $u : \mathbb{R} \to \mathbb{R}$ we have that $\mathbb{E}[\, u(\mathbf{x})\,] \geq \mathbb{E}[\, u(\mathbf{y})\,]$

- Since concavity is a measure of risk-aversion, second-order stochastic dominance helps us to rank distributions by how much risk they involve

mean preserving spreads

- Say that $\mathbf{y}$ is a mean preserving spread of $\mathbf{x}$ if we can write

$$\mathbf{y} = \mathbf{x} + \varepsilon$$

  where $\mathbb{E}[\varepsilon | \mathbf{x}] = 0$
- That is, $\mathbf{y}$ equals $\mathbf{x}$ plus "noise"

Proposition: The following are equivalent

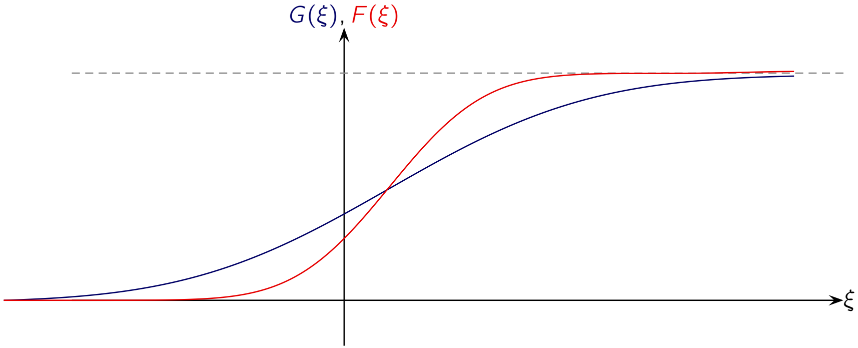
(a) $F \succ_{\text{SOSD}} G$

(b) There exist random variables $\mathbf{x}$ and $\mathbf{y}$ with c.d.f.s F and G, resp., such that $\mathbf{y}$ is a mean preserving spread of $\mathbf{x}$

(c) For every number ξ

$$\int_{-\infty}^{\xi} F(x)\, dx \leq \int_{-\infty}^{\xi} G(y)\, dy$$

second order stochastic dominance



$$F \succ_{\text{SOSD}} G$$